International Journal of Business and Public Management (ISSN: 2223-6244) Vol. 1(1): 66-69
Available online at: http//:www.journals.mku.ac.ke

Full Length Research Paper

Modeling a hierarchical system with double absorbing States

Lydia Musiga[1], John Owino[2], Patrick Weke[3]
[1]School of Mathematics University of Nairobi, Nairobi, Kenya
[2]School of Mathematics University of Nairobi, Nairobi, Kenya
[3]School of Mathematics University of Nairobi, Nairobi, Kenya

Corresponding Author: Lydia Musiga

Recieved: February 24, 2011 Accepted: March 7, 2011

Abstract
This paper describes a Markov Chain transition model for predicting expected numbers of successful graduates and unsuccessful dropouts from an education system, thus the Double absorbing states. This is an improvement on the Single Absorbing State model (see Musiga et al 2010) where successful and unsuccessful dropouts were grouped together. The major limitation of the Single Absorbing State model was the inability to determine the proportions of students who successfully graduated from and unsuccessfully dropped out of the system. The theory of Absorbing Markov Chains is used due to its adaptability to the representation of absorbing rates, transition rates and dropout rates for both successful and unsuccessful students, from an education system. The Double Absorbing States model enables us to predict the numbers of expected qualified personnel vis-à-vis the numbers of unsuccessful dropouts from a system.

Keywords: Hierarchical System, Double Absorbing States, Absorbing Markov Chain, Permanent States, Transition Rates, Dropout Rates
JEL Classification: C30

1.0 Introduction

Previously Musiga *et al* 2010 (2010) modeled a Hierarchical System with a Single Absorbing State for an education system where dropouts and graduates were grouped together. In an education system students either transit from one grade to the next higher grade, repeat the same grade, successfully graduate from the system or drop out of the system before attaining the maximum qualification. Thus students enter permanent states, read absorbing states, either as graduates or dropouts.

In this paper, an undergraduate university degree system is modeled using the Markov Chain approach in which proportions of students who dropout of the system either successfully or unsuccessfully are separately grouped into double absorbing states.

2.0 Literature Review

Stochastic models have been applied in different hierarchical systems. In the educational field, Gani (1963) proposed a Markovian model to forecast enrollment and degrees awarded in Australian Universities. Thonstad (1967) used stochastic models to study enrolments in the Norwegian educational system in his book on educational planning. Uche (1980) applied the Markovian model to the Nigerian educational system.

The Markovian Chain model has been used to study the Kenyan Primary education system see for example Owino (1982) and Odhiambo and Owino (1985). In these studies, several measures of academic survival for the Kenyan Primary education system were considered and compared. Odhiambo and Khogali (1986) studied the Kenyan Primary education system through a cohort analysis where they followed a group of students joining the system at a particular time until the cohort left the system. Owino and Phillips (1988) compared the retention properties of the Kenyan Primary education system between 1964 and 1972 and also between 1972 and 1980. It was found that the system was not time homogeneous in the two time periods. Owino and Odhiambo (1994) used a Markovian model to plan the Kenyan Primary education system by estimating several capital and human resource requirements for the system.

In addition, Mbugua (2005) used the Markov Chain model to estimate the number of new entrants into the Kenyan Primary education system following the introduction of free primary education. Also, a more recent study using the Markov Chain process was based on grade structured control in a manpower system; see Owino and Bodo (2005). In this study, they derived the maintainable grade structures for an academic department.

In the previous paper a Single Absorbing State model was considered. Graduates and dropouts were grouped together. In this paper, graduates and dropouts from an education system are grouped separately into Double Absorbing States.

3.0 The Absorbing State Model

3.1 States in a System

Consider a Markov Chain model with s non-absorbing states; 1, 2, 3...s corresponding to the grades of the system and r absorbing states corresponding to the various final qualifications. Here, $r + s = N$, thus N is the total number of possible states of the system.

An absorbing state is a state which becomes permanent once it has been entered hence transition probabilities between absorbing states should be represented by one, justifying the use of the identity matrix. Transition from an absorbing state to a non-absorbing state which is impossible, should be represented by zero, hence the matrix of zeroes. Transitions from non-absorbing states to absorbing states are possible, likewise transitions between non-absorbing states.

3.2 Transition Probability Matrix

The transition probability matrix P of the Markov chain can then be represented in the following canonical form, assuming time homogeneity, that is

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ . & . & . & \ldots & . & . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 \\ r_{11} & r_{12} & r_{13} & \ldots & r_{1r} & q_{11} & q_{12} & q_{13} & \ldots & q_{1s} \\ r_{21} & r_{22} & r_{23} & \ldots & r_{2r} & q_{21} & q_{22} & q_{23} & \ldots & q_{2s} \\ . & . & . & \ldots & . & . & . & . & \ldots & . \\ r_{s1} & r_{s2} & r_{s3} & \ldots & r_{sr} & q_{s1} & q_{s2} & q_{s3} & \ldots & q_{ss} \end{pmatrix} \quad (1)$$

$$P = \begin{pmatrix} I & O \\ R & Q \end{pmatrix} \quad (2)$$

Where;

I is an $r \times r$ identity matrix which gives transition probabilities between absorbing states

O is an $r \times s$ matrix of zeroes which gives transition probabilities from an absorbing state to a non-absorbing state

$R = (r_{ik})$ is an $s \times r$ matrix, r_{ik} being the probability that a student in grade i at time $(t\text{-}1)$ will graduate with final education k at time t, $i=1, 2,\ldots,s$ and $k = 1, 2,\ldots, r$.

$Q = (q_{ij})$ is an $s \times s$ matrix, q_{ij} being the probability that a student who is in grade i at time $(t\text{-}1)$ will be in grade j at time t; $i, j = 1, 2,\ldots,s$.

Note that:In the double absorbing states model, k takes absorbing state values 1 and 2. Let absorbing state 1 represent dropping out before attaining the maximum qualification and absorbing state 2 represent graduating after attaining the maximum qualification. Hence the R component of matrix P is an $s \times 2$ matrix.

3.3 n-Step Transition Probability Matrix

By the Chapman-Kolmogorov result, the *n*-step transition probability matrix for the process in canonical form is given by

$$P^n = \begin{pmatrix} I & O \\ (I + Q + Q^2 + \ldots + Q^{n-1})R & Q^n \end{pmatrix} \quad (3)$$

that is

$$P^n = \begin{pmatrix} I & O \\ R^n & Q^n \end{pmatrix} \quad (4)$$

Where;

I is an r x r identity matrix which gives transition probabilities between absorbing states in n steps

O is an $r \times s$ matrix of zeroes which gives transition probabilities from absorbing states to non-absorbing states in n steps

$R^n = ((r_{ik}^{(n)})) = (I + Q + Q^2 + \ldots + Q^{n-1})R$ is an $s \times r$ matrix, which gives the probability that a student who is in grade i will graduate with final education k within n years, $I = 1, 2,\ldots,s$ and $k = 1, 2,\ldots,r$. It is also called the completion ratio.

$Q^n = ((q_{ij}^{(n)}))$ is an $s \times s$ matrix which gives the probability that a student who is in grade i will be in grade j, n years later; $i, j = 1, 2,\ldots,s$.

hence

$$\lim(n \to \infty)P^n = \begin{pmatrix} I & O \\ (I - Q)^{-1}R & O \end{pmatrix} \quad (5)$$

The matrix $L = (I - Q)^{-1}$ is called the fundamental matrix of the absorbing Markov Chain.

4 .0 Applications to a Kenyan University Degree Program

4.1 Initial Transition Matrix

Let the states of the system be denoted by integers $1, 2,\ldots, N$ at times $t = 0, 1, 2,\ldots$Let P_{ij} denote the probability that a student in grade i at time $(t\text{-}1)$ will be in grade j at time t, giving rise to transition matrix $P = ((P_{ij}))$; $i.j = 1, 2, \ldots,N$. Let $n_{ij}(t)$ represent the number of students in grade i at time $(t\text{-}1)$ who will be in grade j at time t, also, let $n_i(t\text{-}1)$ represent the number of students in grade i at time $(t\text{-}1)$, then assuming the multinomial distribution, the transition probabilities are estimated from

$$p_{ij} = n_{ij}(t)/n_i(t\text{-}1) \quad (6)$$

where $i, j = 1, 2, \ldots, N$. This is the proportion of students who were in grade i at time $(t\text{-}1)$ who end up being in grade j at time t.

4.2 Initial Transition Process with Double Absorbing States

The data for this study was extracted from Bachelor of Science, Actuarial Science in the School of Mathematics, University of Nairobi. Assuming time homogeneity, students enrolments in grades I, II, III and IV for the year 2004 and enrollments for the same students in grades II, III and IV for the year 2005 were as shown in the following Table1.

The dropout proportions before attaining the maximum qualification for students who were in grades I, II, III and IV were (6/75) = 0.0800, (7/85) = 0.0824, (6/49) = 0.1224 and (1/33) = 0.0304, respectively. Note that the proportion of students who graduated after attaining maximum qualification is (28/33) = 0.8484. This gives rise to the R component of the matrix P.

In the Q component of the matrix P, position 1,1 represents the proportion of students who repeated grade I (2/75) = 0.0267, position 1, 2 represents the proportion of students who proceeded to grade II from grade I (67/75) = 0.8933. The same concept is applied to obtain the relevant proportions of students who were originally in grades II, III and IV for the remaining elements of the Q matrix.

Thus, the transition probability matrix with double absorbing states, P, assuming time homogeneity is;

$$P = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0800 & 0.0000 & 0.0267 & 0.8933 & 0.0000 & 0.0000 \\ 0.0824 & 0.0000 & 0.0000 & 0.0118 & 0.9058 & 0.0000 \\ 0.1224 & 0.0000 & 0.0000 & 0.0000 & 0.0205 & 0.8571 \\ 0.0304 & 0.8484 & 0.0000 & 0.0000 & 0.0000 & 0.1212 \end{pmatrix} \quad (7)$$

4. 3 Completion Rates

4.3.1 Cumulative dropout rate within 'x' years

The dropout rate n years later from grade i is given by

$$r_{ik}^{(n)} = \sum_{j=1}^{s} q_{ij}^{(n-1)} r_{jk} \quad (8)$$

where i, j = 1, ..., s. Note that $q_{ij}^{(n-1)}$ is the probability that a student in grade i will be in grade j,
(n-1) years later and r_{jk} is the probability that a student in grade j at time (t-1) graduates with final education k at time t. Actually, $r_{ik}^{(n)}$ is the $(i, k)^{th}$ element of the product $Q^{(n-1)}R$.

Hence, the cumulative dropout rate within x years from grade i is given

$$r_{ik}^{(x)} = \sum_{n=1}^{x} r_{ik}^{(n)} \quad (9)$$

where, i = 1, ...,s and k = 1,...,r.

Again, $r_{ik}^{(x)}$ is the $(i, k)^{th}$ element of $(I+Q+Q^2+...+Q^{x-1})R$, the basis of computations in this work.

4.4 Completion Rates under Double Absorbing States

Students from grade IV were grouped into those who dropped out of the system before attaining the maximum qualification and those who actually graduated from the system. The completion rate is the $(i, k)^{th}$ element of $(I+Q+Q^2+...+Q^{x-1})R$. Table 2 is a summary of the completion rates within x years using a double absorbing states model.

By the year 2006, 15.57 percent of the students who were in grade I had dropped out of the system before attaining the maximum qualification. In the year 2008, 58.84 percent of the students are expected to graduate from the system. By 2010, a maximum of 22.57 percent will drop out of the system without attaining the maximum qualification. In contrast, within three years, by the year 2007, 65.87 percent of the students are expected to graduate from the system. Eventually, by the year 2011, it is expected that a maximum of 77.43 percent of the students will have graduated from the system. Also, it is expected that by 2009, a maximum of 96.64 per cent of the students who were in grade IV will graduate from the system.

4.5 Absorbing Rates

If students remained in the system indefinitely, then the absorbing rate is given by

$$r_{i1}^{(\infty)} = \sum_{n=1}^{\infty} r_{i1}^{(n)} \quad (10)$$

$$= (I + Q + Q^2 + ...)\, R$$

$$= (I - Q)^{-1}\, R$$

The absorbing rate under double absorbing states is:

$$\begin{pmatrix} 0.2893 & 0.7107 \\ 0.2257 & 0.7743 \\ 0.1552 & 0.8448 \\ 0.0346 & 0.9654 \end{pmatrix}$$

In Table 6.2.1, we considered proportions of students who dropped out of the system without and after attaining the maximum qualification. Considering students who were in grades I, II, III and IV in the long run, 28.93 percent, 22.57 percent, 15.52 percent and 3.46 percent respectively dropped out of the system without attaining maximum qualifications. Considering the same students in the same order, 71.07 percent, 77.43 percent, 84.48 percent and 96.54 percent successfully graduated from the system. These are the very same entries in the absorbing rate under double absorbing states.

5.0 Conclusions

In the single absorbing state case, see Musiga *et al* 2010, all cadres of students were lumped together so it was not possible to determine the proportion of students who successfully graduated from the system. In this double absorbing states case, students who dropped out of the system before attaining the maximum qualification due to various reasons were distinctly separated from those who successfully graduated

from the system. Hence, if our major interest is on the successful graduates from a system, then the double absorbing states model is sufficient.
On the other hand, the double absorbing states model does not give us further insight into the various reasons why and levels at drop out from the system since all the dropouts of different categories are lumped together. This shortcoming has been addressed in the subsequent work.

References

Gani,J., 1963. Formulae for Projecting Enrolments and Degrees awarded in Universities. *Journal of the Royal Statistical Society,* A126, pp. 400-409.

Mbugua,L.N., 2005. Enrolment in Schools due to Free Primary Education, M. Sc. Research Project, submitted to the School of Mathematics, University of Nairobi, Kenya.

Odhiambo, J. W. and A. K. Khogali, 1986. A Transition Model for Estimating Academic Survival through Cohort Analysis. *International Journal of Science and Technology,* Vol. 16, pp. 339-346.

Odhiambo, J. W. and J. O. Owino, 1985. A Stochastic Model for Estimating Academic Survival in an Education System. *Kenya Journal of Science and Technology,* A6, pp. 159-76

Owino, J. O., 1982. Markovian Model for Education Planning, M. Sc. Research Project, submitted to the Department of Mathematics. University of Nairobi, Kenya.

Owino, J. O and J. W. Odhiambo, 1994. A Statistical Model for Planning an Education System. *Discovery and Innovation, Vol. 6, No. 2, pp. 140-144.*

Owino, J. O. and C. M. Philips, 1988. A Comparison of Retention Properties of the Kenyan Primary Education System before and after 1972. *Kenya Journal of Science,* A9, pp. 5-10.

Owino, J. O. and E. O. Bodo, 2005. Maintainability in a Manpower System", East African Journal of Statistics, Vol. 1, No 1, pp. 41-48.

Thornstad, T., 1967. Mathematical Models for Educational Planning, *Paris Organization for Economic Cooperation and Development , pp.* 125-158.

Uche, P. I., 1980. A Transition Model of Academic Survival in a Single Channel System", *International Journal of Education, Science and Technology* 17*,* pp. 177-187.

Appendices

Table 1 Student enrollments 2004/2005

G	E(2004)	PR(2005)	R(2005)	PD(2005)	D(2005)
I	75	67	2	4	2
II	85	77	1	4	3
III	49	42	1	4	2
IV	33	28	4	-	1

Key: *G* represents Grade, *E* represents Enrollment, *PR* represents Proceeded, *R* represents Repeated, *PD* represents Passed and Dropped out and *D* represents Discontinued.

Table 2: Completion Rates Within '*x*' Years Using a Double Absorbing States Model

Years(*x*)	I		II		III		IV	
	D	G	D	G	D	G	D	G
1	0.0800	0.0000	0.0824	0.0000	0.1224	0.0000	0.0304	0.8484
2	0.1557	0.0000	0.1942	0.0000	0.1510	0.7272	0.0341	0.9512
3	0.2577	0.0000	0.2214	0.6587	0.1547	0.8302	0.0345	0.9637
4	0.2847	0.5884	0.2251	0.7598	0.1552	0.8430	0.0346	0.9652
5	0.2887	0.6944	0.2256	0.7726	0.1552	0.8446	0.0346	0.9654
6	0.2892	0.7087	0.2257	0.7741	0.1552	0.8447	0.0346	0.9654
7	0.2893	0.7104	0.2257	0.7743	0.1552	0.8448	0.0346	0.9654
8	0.2893	0.7107	0.2257	0.7743	0.1552	0.8448	0.0346	0.9654
9	0.2893	0.7107	0.2257	0.7743	0.1552	0.8448	0.0346	0.9654
10	0.2893	0.7107	0.2257	0.7743	0.1552	0.8448	0.0346	0.9654

Key: D represents Dropouts and G represents Graduates